

# Ermakov-Lewis angles for one-parameter supersymmetric families of Newtonian free damping modes

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We apply the Ermakov-Lewis procedure to the one-parameter damped modes  $\tilde{y}$  recently introduced by Rosu and Reyes, which are related to the common Newtonian free damping modes  $y$  by the general Riccati solution [H.C. Rosu and M. Reyes, Phys. Rev. E **57**, 4850 (1998)]. In particular, we calculate and plot the angle quantities of this approach that can help to distinguish these modes from the common  $y$  modes.

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In a previous paper hereafter denoted as I [1], the non-uniqueness of the factorization of linear second-order differential operators has been exploited on the example of the classical Newtonian free damped oscillator, i.e.

$$Ny \equiv \left( \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2 \right) y = 0. \quad (1)$$

The coefficient  $2\beta$  is the friction constant per unit mass and  $\omega_0$  is the natural frequency of the oscillator (SI units assumed all over the work).

The more general supersymmetric partner equation

$$\tilde{N}_g \tilde{y} \equiv \left( \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2 - \frac{2\gamma^2}{(\gamma t + 1)^2} \right) \tilde{y} = 0 \quad (2)$$

has been obtained in I. This new second-order linear damping equation contains the additional last term with respect to its initial partner (1), which may be thought of as the general Darboux transform part of the frequency [4].  $T = 1/\gamma$  occurs as a new time scale in the Newtonian damping problem. If this time scale is infinite, the ordinary free damping is recovered unless for the critical case which is special even in ordinary damping. As explained in I, the  $\tilde{y}$  modes can be obtained from the  $y$  modes by operatorial means. In the following we shall call them  $\gamma$  modes. For the three types of free damping they have been obtained in I as follows:

(i) For underdamping,  $\beta^2 < \omega_0^2$ , denoting  $\omega_u = \sqrt{\omega_0^2 - \beta^2}$  the underdamped  $\gamma$  modes are

$$\tilde{y}_u = -\tilde{A}_u e^{-\beta t} \left[ \omega_u \sin(\omega_u t + \phi) + \frac{\gamma}{\gamma t + 1} \cos(\omega_u t + \phi) \right]. \quad (3)$$

(ii) For overdamping,  $\beta^2 > \omega_0^2$  and  $\omega_o = \sqrt{\beta^2 - \omega_0^2}$ , the overdamped  $\gamma$  modes are

$$\tilde{y}_o = -\tilde{A}_o e^{-\beta t} \left[ \omega_o \sinh(\omega_o t + \phi) - \frac{\gamma}{\gamma t + 1} \cosh(\omega_o t + \phi) \right]. \quad (4)$$

(iii) For critical damping,  $\beta^2 = \omega_0^2$ . The critical  $\gamma$  solutions are given by

$$\tilde{y}_c = \left[ \frac{-A_c \gamma}{\gamma t + 1} + \frac{D_c}{\gamma^2} (\gamma t + 1)^2 \right] e^{-\beta t}. \quad (5)$$

These are the *only* possible types of one-parameter damping modes related to the free damping ones by means of Witten's supersymmetric scheme [2] and the general Riccati solution [3].

In practice the new parameter  $\gamma$  can be very close to zero. In this case, it is very difficult to differentiate the  $\gamma$  modes from the ordinary ones. The only means we can think of is by recording somehow the geometric angle associated to the  $\gamma$  modes and compare it with the same quantity in the ordinary damping cases. One is led to this conclusion noticing that the  $\gamma$  modes have time-dependent frequencies  $\omega^2(t) = \omega_0^2 - \frac{2\gamma^2}{(\gamma t + 1)^2}$  and hence for them the Ermakov-Lewis (EL) procedure can be naturally applied [5] (for a recent review, see [6]). For  $\omega_0 \neq \beta$ , Eq. (2) can be reduced to a Bessel equation and the solutions can be written as follows

$$\Psi_u = \tau^{1/2} \left[ A J_{\frac{3}{2}}(k\tau) + B Y_{\frac{3}{2}}(k\tau) \right] e^{-\beta\tau} \quad (6)$$

and

$$\Psi_o = \tau^{1/2} \left[ C I_{\frac{3}{2}}(k\tau) + D K_{\frac{3}{2}}(k\tau) \right] e^{-\beta\tau}, \quad (7)$$

where  $\tau = \gamma t + 1$  and  $k^2 = \frac{\omega_0^2 - \beta^2}{\gamma^2}$ . When  $k \rightarrow \infty$  (i.e.,  $\gamma \rightarrow 0$ ), we can do Hankel's asymptotic expansions, i.e., of large Bessel argument but fixed Bessel order (we shall not reproduce these formulas here, the reader is directed to Abramowitz and Stegun [7]). The point is that one is indeed able to get the solutions obtained by operatorial means from inspecting Hankel's expansions. Thus, the supersymmetric operatorial procedure gives merely the asymptotic  $\gamma \rightarrow 0$  solutions, which however could be the most relevant from the physical viewpoint in this context.

In the EL approach the angular quantities are given by the following formulas [6,10]

$$\Delta\theta^d = \int_0^T \left[ \frac{e^{-2\beta t'}}{\rho^2} - \frac{1}{2} \frac{d}{dt'} (e^{2\beta t'} \dot{\rho} \rho) + e^{2\beta t'} \dot{\rho}^2 \right] dt' \quad (8)$$

and

$$\Delta\theta^g = \frac{1}{2} \int_0^T \left[ \frac{d}{dt'} (e^{2\beta t'} \dot{\rho} \rho) - 2e^{2\beta t'} \dot{\rho}^2 \right] dt', \quad (9)$$

for the dynamical and geometrical angles, respectively. Thus, the total angle will be

$$\Delta\theta^t = \int_0^T \frac{e^{-2\beta t'}}{\rho^2} dt' . \quad (10)$$

The so-called Pinney function  $\rho$  is the solution of Pinney's nonlinear equation [8]

$$\rho''(t) + p(t)\rho'(t) + q(t)\rho = \frac{C}{\rho^3(t)} \exp\left(-2 \int^t p(t')dt'\right) \quad (11)$$

for  $C = \text{constant} (=1)$ ,  $p(t) = 2\beta$  and  $q(t) = \omega_0^2 - \frac{2\gamma^2}{(\gamma t + 1)^2}$ . For  $\rho \neq \text{constant}$  there is a definite prescription of calculating  $\rho$  in terms of two independent solutions of the corresponding linear equation. We have followed the method of Eliezer and Gray [9] for  $\rho(t)$  in terms of linear combinations of the aforementioned Bessel functions (for  $A = B = C = D = 1$ ) that satisfy the initial conditions as given by those authors. In the critical damping case, we used the modes of Eq. (5) with  $A_c = D_c = 1$ . The results of the calculations for some particular values of the parameters are plotted in Figs. 1a,b,c, 2a,b,c, 3a,b,c for the  $\gamma$  underdamped, overdamped, and critical cases, respectively. For comparison, the angle quantities for  $\gamma = 0$ , within the same calculational scheme, are displayed in Figs. 1a',b',c', 2a',b',c', 3a',b',c', respectfully.

### ACKNOWLEDGMENT

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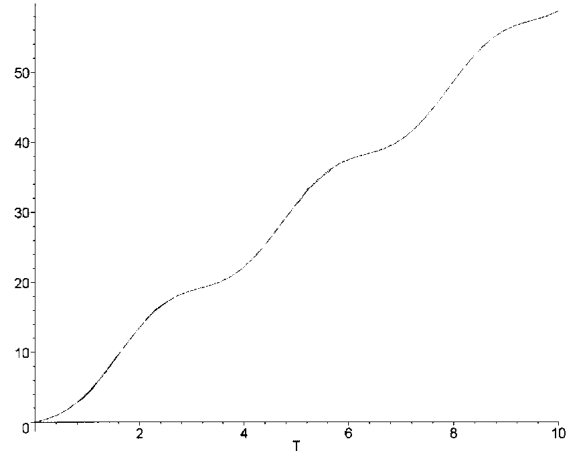


Fig. 1a

The dynamical angle in the underdamped case for the following set of parameters:  $\omega_0 = \sqrt{2}$ ,  $\beta = 1$ ,  $\gamma = 0.1$ .

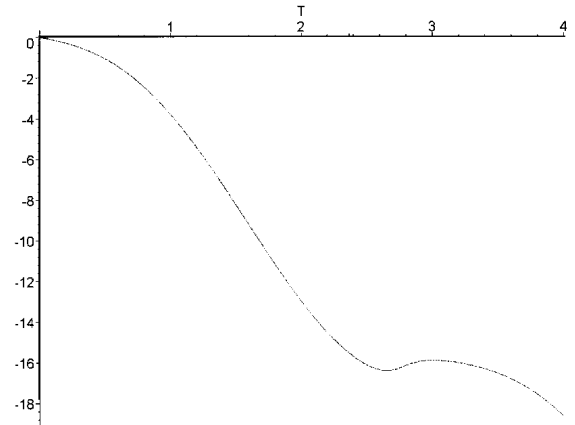


Fig. 1b

The geometric angle in the underdamped case and the same parameters.

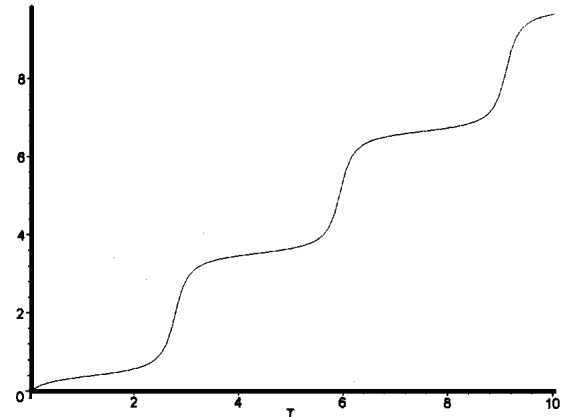


Fig. 1c

The total angle in the underdamped case and the same parameters.

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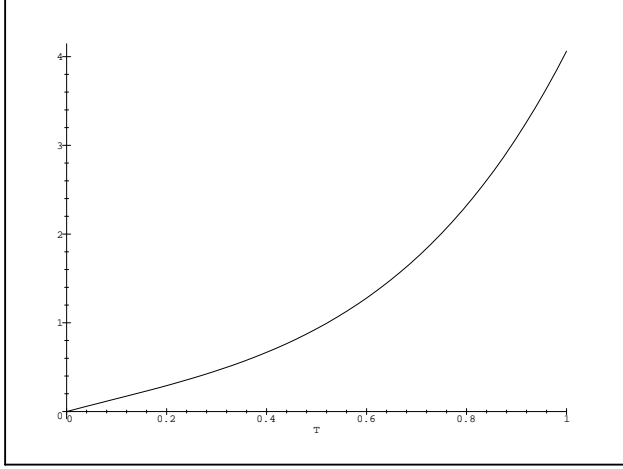


Fig. 2a

The dynamical angle in the overdamped case for  $\omega_0 = 1$ ,  
 $\beta = \sqrt{2}$ ,  $\gamma = 0.1$ .

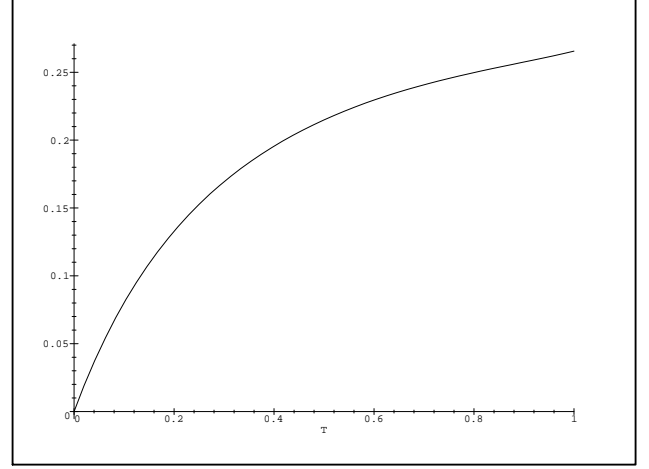


Fig. 2c

The total angle in the overdamping case for the same  
parameters.

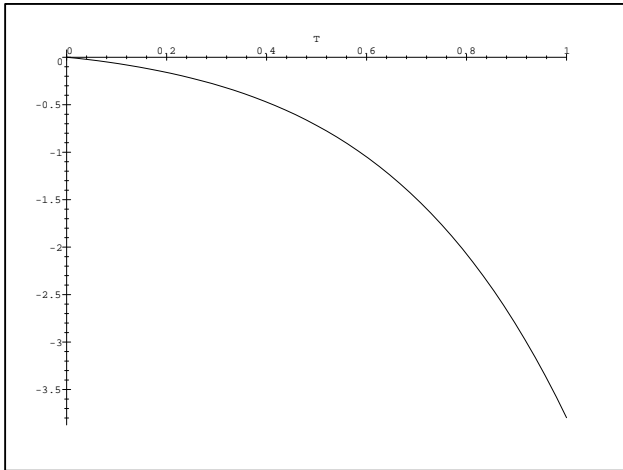


Fig. 2b

The geometric angle in the overdamping case for the same  
parameters.

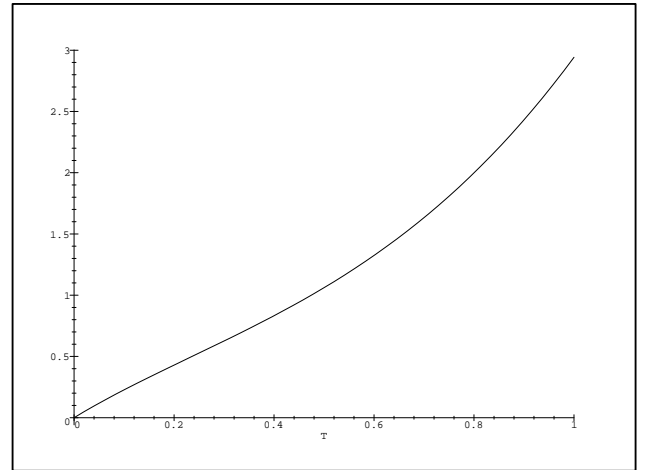


Fig. 3a

The dynamical angle in the critical case for  $\omega_0 = \beta = 1$  and  
 $\gamma = 0.1$ .

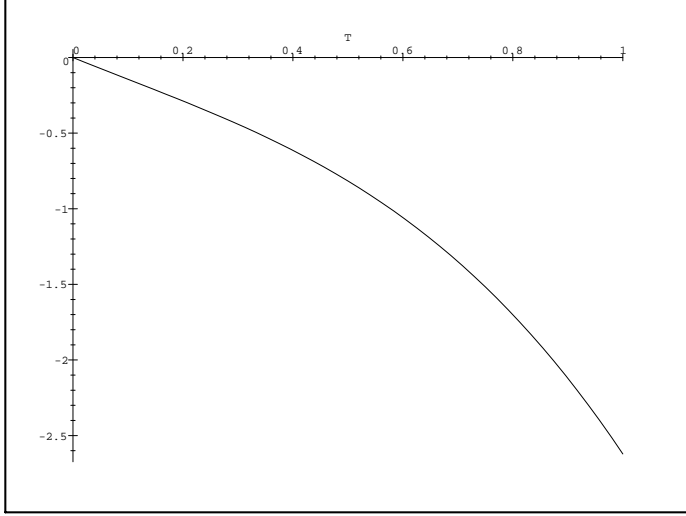


Fig. 3b  
The geometrical angle in the critical case for the same parameters.

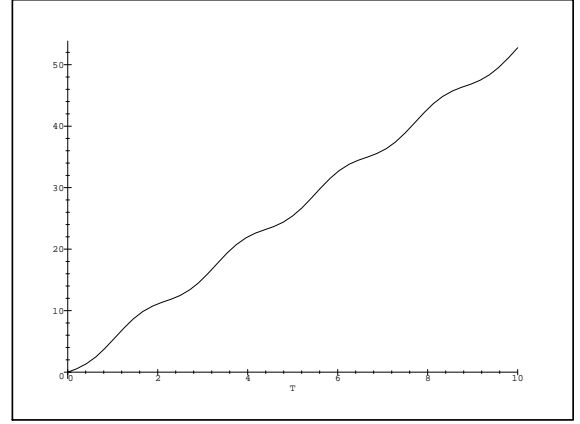


Fig. 1a'  
The dynamical angle in the underdamped case for the same  $\omega_0, \beta$  parameters as in Fig. 1a and  $\gamma = 0$ .

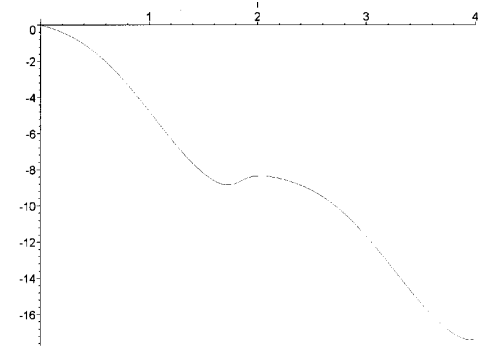


Fig. 1b'  
The geometrical angle in the underdamped case for the same  $\omega_0, \beta$  parameters and  $\gamma = 0$ .

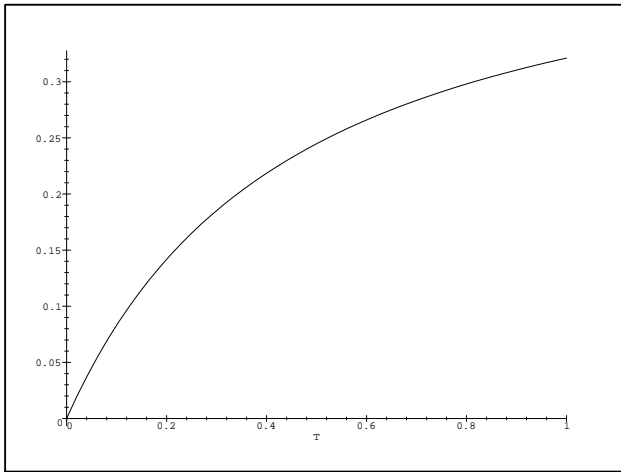


Fig. 3c  
The total angle in the critical case for the same parameters.

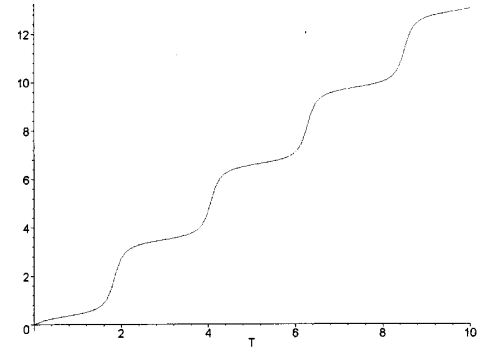


Fig. 1c'  
The total angle in the underdamped case for the same  $\omega_0, \beta$  parameters and  $\gamma = 0$ .

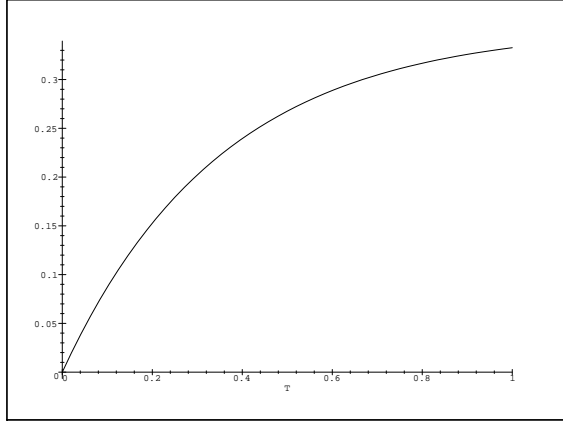


Fig. 2a'

The dynamical angle in the overdamped case for the same  $\omega_0, \beta$  parameters as in Fig. 2a and  $\gamma = 0$ .

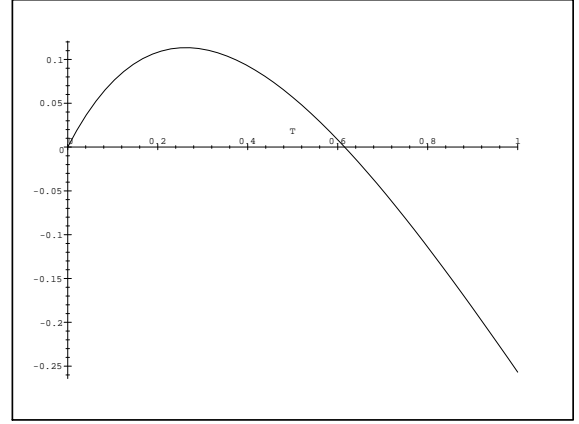


Fig. 2c'

The total angle in the overdamped case for the same  $\omega_0, \beta$  parameters and  $\gamma = 0$ .

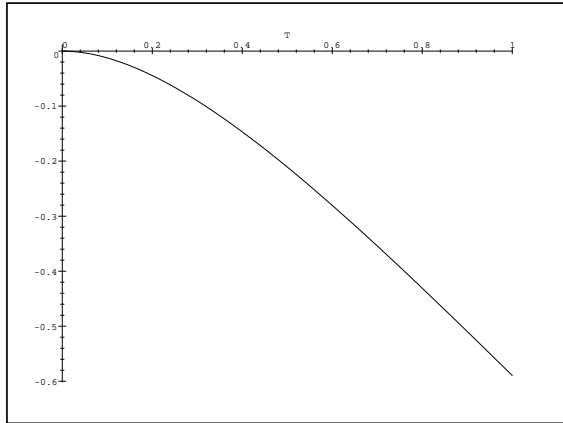


Fig. 2b'

The geometrical angle in the overdamped case for the same  $\omega_0, \beta$  parameters and  $\gamma = 0$ .

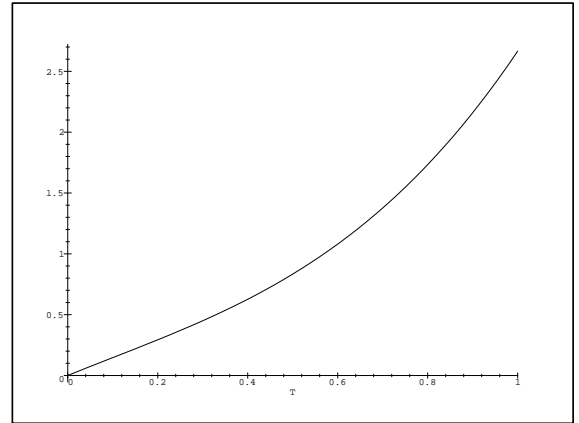


Fig. 3a'

The dynamical angle in the critical case for  $\omega_0 = \beta = 1$  and  $\gamma = 0$ .

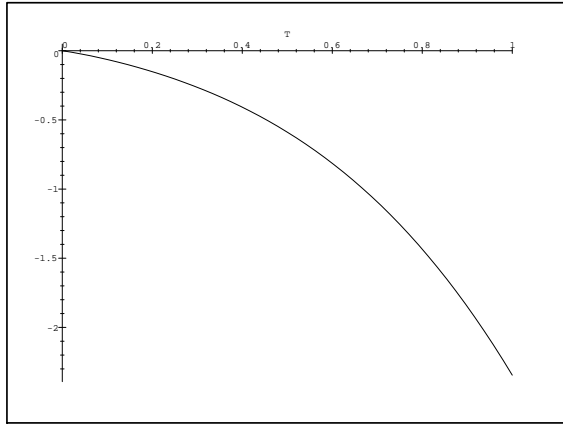


Fig. 3b'

The geometrical angle in the critical case for  $\omega_0 = \beta = 1$  and  $\gamma = 0$ .

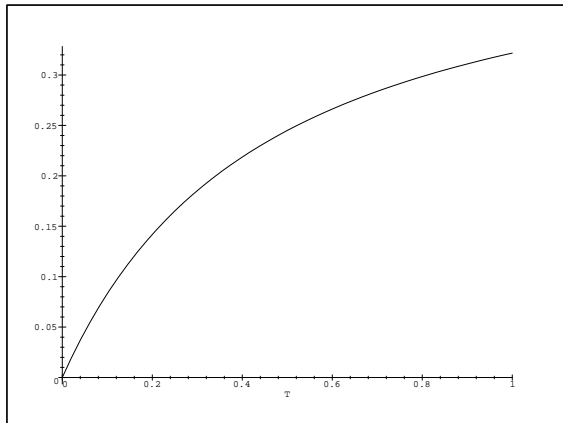


Fig. 3c'

The total angle in the critical case for  $\omega_0 = \beta = 1$  and  $\gamma = 0$ .